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Some Statistical Aspects of the Determination of a Safe Life from Fatigue Data

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SOME STATISTICAL ASPECTS OF THE DETERMINATION OF A SAFE LIFE FROM FATIGUE DATA

by

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SUMMARY

The probability that within a future large second sample no failures will occur before the expiration of a safe service life estimated from a small first sample and the probability that the proportion of all future observations failing before the estimated safe service life is smaller than a given proportion, are the two measures of safety that we adopt here.

Assuming the logarithm of the fatigue life is normal with known variance, we derive formulae for these measures of safety. Setting the safe life as some fraction of the mean estimated by the first sample, we then compare the influence of other parameters on these measures of safety.

From this assumption it is shown that one has virtually as high an assurance of safety, measured by the first criterion, when using only the minimum of the first sample, as one does by using all the observations in the first sample. If one uses the standard second criterion, namely, the confidence level of a lower tolerance bound, as a measure such an advantage is not retained.

ACKNOWLEDGMENT

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1. INTRODUCTION

Suppose we have a structural component in an airplane which is subject to failure from fatigue, perhaps due to the cyclic loading of the ground-air-ground cycle, to acoustic loading, or to any other well-defined phenomenon. We ask what estimate we can make of a safe service life, or more specifically, the lower quantities of the distribution of the time until failure, from a first sample such as the simulated testing of a small number of specimens, or from the observation of the first few units of production.

The important question with which we concern ourselves is, what is the probability that the weakest component within a fleet of airplanes to be produced, or coming into service, will fail before a certain time established by derating the life estimated from the first sample. The problem is to determine the derating procedure so there is little probability of failure within the future fleet.

We let $X_{i,m}$ $i=1,\ldots,m$ be the i^{th} failure time resulting from the first sample, and we let $Y_{i,n}$ $i=1,\ldots n$ be the i^{th} failure time from the second sample of components in service. We assume that the initial sample of lives, simulated in the laboratory, or the first service lives have the same distribution as the service lives later on, the logarithm of which we assume is normally distributed. We label the common distribution of life with F.

From the first sample taken, we have the data $\underset{\sim}{X} = (X_{1,m},\ldots,X_{m,m})$ and a given derating function d which we use to obtain the derated (safe) life $d(\underset{\sim}{X})$. The probability that no failures will occur in the fleet before the safe life we call the fleet assurance, labeled α; it is given by

$$(1.1) \qquad \alpha = P[Y_{1,n} > d(\underset{\sim}{X})] = \int_{\underset{\sim}{X}} \{1-F[d(\underset{\sim}{x})]\}^{n} \, dG(\underset{\sim}{x})$$

where G is the joint distribution of the sample $\underset{\sim}{X}$.

The probability that a proportion of at least β of all future observations will not fail before the derated life we call the *confidence*, labeled γ, which is

$$(1.2) \qquad \gamma = P\{1-F[d(\underset{\sim}{X})] > \beta\}.$$

Clearly in this case the derated life is just a lower tolerance bound for the population described by the distribution F, and the fleet size n does not enter in.

We chose the function d to be some fraction of an estimate of the mean. Our problem is to study the influence of various parameters, such as sample size and variance, on the measures of safety, while keeping in mind the economic desirability of using as few of the first few ordered observations as possible.

2. EVALUATION OF THE SAFETY FACTORS USING FIRST ORDERED OBSERVATIONS

We assume throughout that the logarithm of the fatigue life is normally distributed with unknown mean μ and known standard deviation σ, and the derating function is some fraction of the estimated mean life. Hence from (1.1) we have

(2.0.1) $$\alpha = P[Y_{1,n} > p\hat{\nu}] \quad , \text{ for some } 0 < p < 1,$$

where $\nu = \exp\{\mu + \sigma^2/2\}$ is the mean (expected) life of a component and $\hat{\nu}$ is the estimate of ν made from $\underset{\sim}{X}$.

A current procedure is to take a small sample from simulated laboratory testing, perhaps $m = 2$ or $m = 3$, and set

(2.0.2) $$\hat{\mu} = \frac{1}{m} \sum_{1}^{m} \ell n \, X_{i,m}.$$

As we shall prove later on, in this case, we have for the two safety indices α, γ

(2.0.3) $$\alpha = \int_{-\infty}^{\infty} \Phi^n(\xi + \frac{x}{\sqrt{m}}) \, d\Phi(x)$$

(2.0.4) $$\gamma = \Phi[\sqrt{m} \, (\xi - \zeta_\beta)],$$

where

(2.0.4.1) $$- \xi = \frac{\ell n \, p}{\sigma} + \frac{\sigma}{2}, \qquad \Phi(\zeta_\beta) = \beta$$

with Φ the cumulative distribution function of the standard normal, defined by

(2.0.5) $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left|- \frac{t^2}{2}\right| dt.$$

Notice that neither α nor γ depend upon μ, and that as m grows large α approaches the value $\Phi^n(\xi)$, and γ increases to the value one if $\xi > \zeta_\beta$ and decreases to zero if $\xi < \zeta_\beta$. To obtain some idea of the magnitude of these indices we take $n = 200$, $p = \frac{1}{4}$, $\sigma = .46$ (this last is equivalent to taking the standard deviation of the common logarithm of life to be equal to .2). We then have $\xi = 2.95$, and from table II with $m = 3$, we see that $\alpha = .64$, and by taking $\zeta_\beta = 2.326$ (i.e. $\beta = .99$) we can compute that $\gamma = .86$ from tables of the standard cumulative normal distribution.

A natural question to ask is: If we increase m and observe only the first of the ordered observations (or perhaps the first few), can we increase α or γ? If the answer is yes, by how much?

In order to obtain as much generality as possible, we consider observing only $X_{1,m},\ldots,X_{r,m}$ where r may be 1 up to m. Thus if $r = m$, we have the entire sample. If $r = 1$ we observe only the life of the weakest component out of the m simulated lives.

Following our basic assumption, we let

$$Z_{i,m} = \ln X_{i,m} \qquad i = 1,\ldots,r$$

Now we state

<u>THEOREM 1</u>. *If $Z_{1,m},\ldots,Z_{r,m}$ are the first r of m ordered independent normal observations with unknown mean μ but known variance σ^2, the maximum likelihood estimate $\hat{\mu}$ of the mean is*

(2.1) $$\hat{\mu} = Z_{r,m} + \sigma \cdot \hat{\tau}_{r,m}$$

where $\hat{\tau}_{r,m}$ *is the solution of* [2.1.3].

Proof. The first r of m ordered normal observations have the joint density on $-\infty < z_1 < \cdots < z_r < \infty$.

(2.1.1) $$\frac{m!}{(m-r)!}\left[1-\Phi\left(\frac{z_r-\mu}{\sigma}\right)\right]^{m-r}\left[\prod_{i=1}^{r}\frac{1}{\sigma}\phi\left(\frac{z_i-\mu}{\sigma}\right)\right],$$

where ϕ is the density of (2.0.5).

Then letting K be a constant independent of μ, and L be the logarithm of the joint density (i.e., the log-likelihood), we have

$$L = K - \sum_1^r \tfrac{1}{2}\left(\frac{z_i-\mu}{\sigma}\right)^2 + (m-r)\ln\Phi\left(\frac{\mu-z_r}{\sigma}\right)$$

(2.1.2) $$\sigma\frac{\partial L}{\partial\mu} = \sum_1^r\left(\frac{z_i-\mu}{\sigma}\right) + \frac{(m-r)}{\Phi\left(\frac{\mu-z_r}{\sigma}\right)}\ \phi\left(\frac{\mu-z_r}{\sigma}\right) = 0$$

Letting $\psi(x) = \phi(x)/\Phi(x)$, $\tau = (\mu-z_n)/\sigma$ and by adding and subtracting z_r in the summation we see (2.1.2) is equivalent with

(2.1.3) $$(m-r)\ \psi(\tau) - r\tau = \sum_{i=1}^{r}\frac{z_r-z_i}{\sigma}\ .$$

We call the unique solution $\hat{\tau}_{r,m}$ which exists by the decreasing monotonicity of ψ.

If the graph of $\psi(\tau)$, as given in figure [1], is used, an approximate solution to (2.1.3) can be readily found. The tables of reference [2], can be used to determine the root more accurately.

Now we state the

COROLLARY 1. *If* $r = 1$ *then* $\hat{\tau}_{1,m}$ *is a constant depending only upon* m *and is the solution, call it* τ_m, *of the equation in* t

$$\frac{\psi(t)}{t} = \frac{1}{m-1}$$

and

$$\hat{\mu} = Z_{1,m} + \sigma \cdot \tau_m.$$

We present a table of values of τ_m in table I. And of course, we can obtain the result of (2.0.2):

COROLLARY 2. *If* $r = m$, *then* $\hat{\tau}_{m,m} = -\frac{1}{m\sigma}\sum_{i=1}^{m}(Z_{m,m}-Z_{i,m})$

and

$$\hat{\mu} = Z_{m,m} + \sigma \cdot \hat{\tau}_{m,m} = \frac{1}{m}\sum_{1}^{m} Z_{i,m}.$$

It follows from (2.0.1) that

$$\alpha = P[\ell nY_{1,n} > \ell np + \hat{\mu} + \sigma^2/2]$$

$$(2.2.1) \qquad \alpha = P[\ell n Y_{1,m} > \ell np + Z_{r,m} + \sigma \cdot \hat{\tau}_{r,m} + \sigma^2/2].$$

Letting ξ be as defined in (2.0.4.1) and setting

$$(2.2.2) \qquad U_{1,n} = \frac{1}{\sigma} \ell n Y_{1,n}, \quad V_{i,m} = \frac{1}{\sigma} Z_{i,m} \qquad i = 1,\ldots,m$$

we have, letting $\Lambda_{r,m}(\tau)$ be the left hand side of (2.1.3),

$$(2.2.3) \qquad \alpha = P\left[U_{1,n} > V_{r,m} - \xi + \Lambda_{r,m}^{-1}\left[\sum_1^r (V_{r,m} - V_{i,m})\right]\right]$$

which we can express in integral form as

$$(2.3) \quad \alpha = \int_{-\infty<v_1<\cdots<v_r<\infty} \int \Phi^n\left[\xi - v_r - \Lambda_{r,m}^{-1}\left[\sum_1^r (v_r - v_i)\right]\right] h(v_1,\cdots,v_r) \, \Pi dv_i$$

where $h(v_1,\ldots,v_r)$ is the joint density in (2.1.1) with $\sigma = 1$.

Unfortunately this integral is not easily simplified in the general case. But to obtain some idea of its behavior, we give the explicit expressions for the two extreme cases $r = 1$ and $r = m$.

For $r = 1$ we have from (2.2.3)

$$\begin{aligned} \alpha &= P[U_{1,n} > V_{1,m} - \xi + \tau_m] \\ &= m \int_{-\infty}^{\infty} \Phi^n(\xi - v - \tau_m) \, [1-\Phi(v)]^{m-1} \phi(v) dv. \end{aligned}$$

By using the fact that ϕ is even we obtain a function of ξ, say,

$$A_1(\xi) = m \int_{-\infty}^{\infty} \Phi^n[\xi - \tau_m + x] \Phi^{m-1}(x) \phi(x) dx.$$

For $r = m$, we have from (2.2.3)

$$\alpha = P\left[U_{1,n} > -\xi + \frac{1}{m}\sum_{1}^{m} V_{i,m}\right]$$

and in the same manner as above we have a function of ξ, call it

$$A_2(\xi) = \int_{-\infty}^{\infty} \Phi^n\left[\xi + \frac{x}{\sqrt{m}}\right]\phi(x)\,dx.$$

which we recognize as that given in (2.0.3).

The comparison of the values of A_1 and A_2 for $n = 200$, $m = 5$, is given in figure 2. We note that for a given value of ξ $A_1(\xi) > A_2(\xi)$ for $\xi < 3$, but the reverse inequality holds for $\xi > 3$. However, there is not much difference considering that we use only the minimum observation.

A tabulation of $A_2(\xi)$ is given in table II for m and n. As a simplification we also present a tabulation not of $A_1(\xi)$ for various values of m and n but of

(2.4) $$A_3(\rho) = \int_{-\infty}^{\infty} \Phi^n(\rho+x)\,d\Phi^m(x)$$

where $\rho = \xi - \tau_m$.

A short tabulation of $A_3(\rho)$ is given in table III. This accomplishes our study of fleet assurance.

Now we turn to the question of confidence in these two cases. By definition (1.2)

$$\gamma = P[F(p\hat{\nu}) < 1-\beta]$$

where F is here the log-normal distribution with parameters μ and σ^2.

Now

$$F(p\hat{\nu}) = \Phi\left(\frac{\ell np + \hat{\mu} + (\sigma^2/2) - \mu}{\sigma}\right).$$

Hence by letting ξ and ζ_β be as defined in (2.0.4.1)

(2.5) $$\gamma = P\left[\frac{\hat{\mu}-\mu}{\sigma} < \xi + \zeta_{1-\beta}\right]$$

$$= P\left\{V_{r,m} + \Lambda_{r,m}^{-1}\left[\sum_{1}^{r}(V_{r,m}-V_{i,m})\right] < \xi + \zeta_{1-\beta}\right\}$$

This can also be expressed as an integral but we give only the two cases $r = 1$, $r = m$.

For $r = 1$

$$\gamma = P[V_{1,m} < \xi - \tau_m - \zeta_\beta] = 1 - [1 - \Phi(\xi - \tau_m - \zeta_\beta)]^m$$

(2.6) $$\gamma = 1 - \Phi^m(\tau_m + \zeta_\beta - \xi)$$

For $r = m$ we obtain the result stated in (2.0.4).

(2.7) $$\gamma = \Phi[\sqrt{m}(\xi - \zeta_\beta)]$$

Again, for purposes of comparison, we take the two functions

$$B_1(\xi) = 1 - \Phi^m(\tau_m - \xi + \zeta_\beta)$$

$$B_2(\xi) = \Phi[\sqrt{m}(\xi - \zeta_\beta)].$$

which we present in figure 3 for $m = 5$, $\beta = .99$.

It is clear that $B_2(\xi)$ is to be preferred, it being the greater for almost all values of ξ.

3. SAFETY FACTORS IN THE CASE SOME INITIAL OBSERVATIONS ARE LOST

In order to obtain a bit more generality we consider the estimation problem in the case when the first few operations may not have been observed. This could, for example, be the case when the phenomenon under consideration is the occurrence of a fatigue crack of a specified dimension, and at a certain time regular inspection reveals that several of these have already occurred. Hence we consider observing only $X_{s,m},\ldots,X_{r,m}$ where $1 \le s \le r \le m$. Thus if $s = 1$, $r = m$ we have the entire first sample. If $r = s = 1$ we observe only the life of the weakest component of the first sample.

<u>THEOREM 2</u>. *If* $Z_{s,m},\ldots,Z_{r,m}$ *for* $1 \le s \le m$ *are the* $(r+1-s)$ *ordered observations of* m *independent normal observations with unknown mean but known variance* σ^2, *the maximum likelihood estimate* $\hat{\mu}$ *of the mean is*

$$\hat{\mu} = Z_{r,m} + \sigma\cdot\hat{T}(s,r,m)$$

where $\hat{T}(s,r,m)$ *is the solution of equation (3.1).*

<u>Proof</u>. Proceeding exactly as before the s^{th} through the r^{th} ordered observations have the joint density on $\infty < z_s < \cdots < z_r < \infty$

$$\frac{m!}{(m-r)!(s-1)!}\,\Phi^{s-1}\left(\frac{z_s-\mu}{\sigma}\right)\left[\prod_{i=s}^{r}\frac{1}{\sigma}\,\phi\left(\frac{z_i-\mu}{\sigma}\right)\right]\Phi^{m-r}\left(\frac{\mu-z_r}{\sigma}\right).$$

Using the notation of Theorem 1 we obtain

$$\text{(3.1)}\qquad \sigma\,\frac{\partial L}{\partial\mu} = \sum_{i=s}^{r}\left(\frac{z_i-\mu}{\sigma}\right) - (s-1)\,\psi\left(\frac{z_s-\mu}{\sigma}\right) + (m-r)\,\psi\left(\frac{\mu-z_r}{\sigma}\right) = 0.$$

which is equivalent with

$$(3.2)\qquad (m-r)\psi(\tau) - (s-1)\psi\left(\frac{z_s-z_r}{\sigma} - \tau\right) - (r-s+1)\tau = \sum_{i=s}^{r}\left(\frac{z_r-z_i}{\sigma}\right).$$

We call the solution of (3.2) $\hat{T}(s,r,m)$, which exists uniquely since the left hand side is a decreasing function of τ.||

Also, note that $\hat{T}(1,r,m) = \hat{\tau}_{r,m}$.

As before, the graph of ψ as given in figure [1] can be used to obtain an approximate solution to (3.2) and the tables of reference [2] can be utilized to obtain the root more accurately.

The problem of obtaining a simple formula for the assurance or confidence in this general case seems rather difficult. But there follows immediately from Theorem 2 the

<u>COROLLARY 3</u>. *If $s = r$ then $\hat{T}(r,r,m)$ is a constant which depends only upon r,m, call it $t_{r,m}$, which is the solution of the equation in t*

$$(m-r)\psi(t) - (r-1)\psi(-t) - t = 0$$

and

$$(3.3)\qquad \hat{\mu} = Z_{r,m} + \sigma\cdot t_{r,m}.$$

Now if we want to obtain an expression for the assurance, we have directly from (2.2.1) by replacing $\hat{\tau}_{r,m}$ by $t_{r,m}$ and using the notation of (2.2.2)

$$\alpha = P[U_{1,n} > -\xi+t_{r,m}+V_{r,m}]$$

Since $V_{r,m}$ has the density defined by

$$r\binom{m}{r}\Phi^{r-1}(v)\ [1-\Phi(v)]^{m-r}\phi(v) \qquad -\infty < v < \infty$$

we obtain

$$(3.4)\quad \alpha = r\binom{m}{r}\int_{-\infty}^{\infty}\Phi^{n}(\xi-t_{r,m}+x)\ \Phi^{r-1}(-x)\ \Phi^{m-r}(x)\ \phi(x)dx.$$

A table of this integral would involve three parameters so it would seem more practical to reserve computation for those cases of specific interest. No tabulation is made.

For the confidence in this case we have from equation (2.5)

$$\gamma = P[(Z_{r,m}+\sigma\cdot t_{r,m}-\mu)/\sigma < \xi + \zeta_{1-\beta}]$$

$$\gamma = P[V_{r,m} < \xi - t_{r,m} + \zeta_{1-\beta}].$$

But since the distribution of $V_{r,m}$ is given by

$$P[V_{r,m} < x] = 1 - \sum_{j=0}^{r-1}\binom{m}{j}\Phi^{j}(x)\ [1-\Phi(x)]^{m-j}$$

$$= \sum_{j=0}^{m-r}\binom{m}{j}\ \Phi^{j}(-x)\ \Phi^{m-j}(x)$$

we can obtain values of the confidence γ by using first tables of Φ (reference [2]) and then the binomial tables (reference [1]) to calculate it.

We call attention to the further specialization which occurs when the inspection reveals only the strongest item.

<u>COROLLARY 4</u>. *If* $s = r = m$ *then in (3.3),* $t_{m,m} = -\tau_m$ *where* τ_m *was defined in Corollary 1 and given in table 1.*

In this special case

$$\alpha = m\int_{-\infty}^{\infty} \Phi^{n}(\xi+\tau_m-x)\Phi^{m-1}(x)\phi(x)\,dx$$

$$\gamma = P[V_{m,m} \leq \xi+\tau_m+\zeta_{1-\beta}]$$

$$\gamma = \Phi^{m}[\xi+\tau_m+\zeta_{1-\beta}].$$

4. NUMERICAL COMPARISONS

We make the assumption throughout that $\sigma = .46$. This is equivalent to taking the standard deviation of the common logarithm of the life to be equal to .2.

Let us see, if we derate by fraction $p = \frac{1}{4}$ and we have a fleet size of $n = 200$, whether we have more assurance by choosing the minimum of five observations or whether we have more assurance by using all three observations.

Take $m = 5$, $r = 1$, $p = \frac{1}{4}$; then $\xi = 2.95$, and for $n = 200$ we find from table I that $A_1(2.95) = .66$, which is about twice the value of $A_2(2.95)$ for $m = 3$, which we discussed on page 4.

Suppose we want the assurance to be .90 when we use the minimum of five observations. We ask what value of p should we take? From table III we see that if $A_3(\rho) = .90$ then $\rho = 2.6$ since $\rho = \xi - \tau_5$, $\xi = 3.66$ by table I. Now from (2.0.4.1) we have

$$p = \exp\{-\sigma\xi - \sigma^2/2\} \tag{4.1}$$

and since $\sigma = .46$

$$p = e^{-1.79} = .16 = \frac{1}{6}$$

To show the sensitivity of the assurance to σ, let us suppose that $\sigma = .2$, instead of .46. From equation (4.1) above we obtain

$$p = e^{-.75} = .47 = \frac{1}{2} .$$

The tables which are provided are of course very sketchy and are included only to give an idea of the magnitude of the functions A_1 and A_2 for a few selected values of n and m. However, the programs which we used to compute these functions are available by writing to the author, and would be provided upon request.

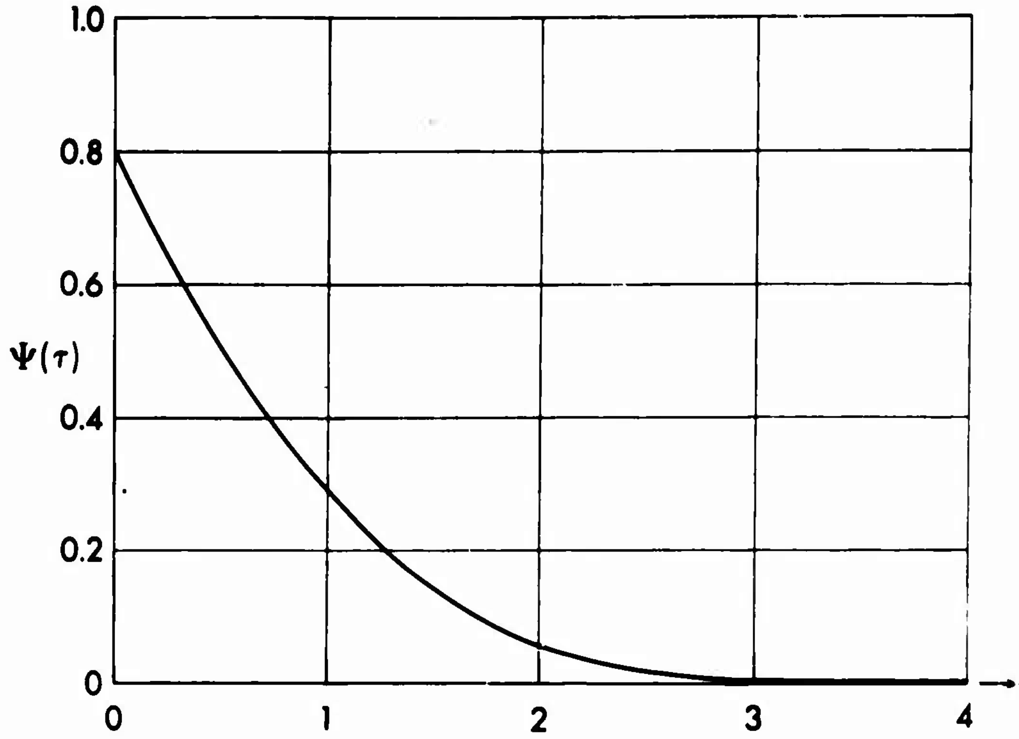

Figure 1a

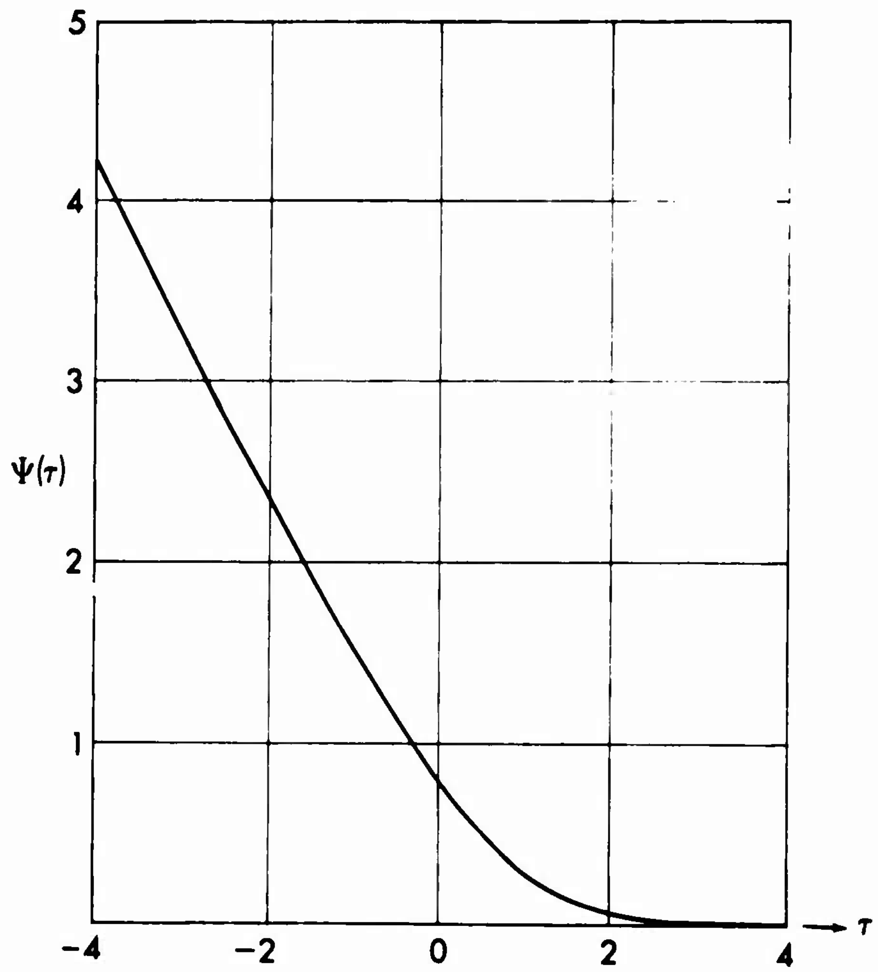

Figure 1b

Graphs for obtaining the solution of the maximum likelihood estimate of the mean when only the weakest component is obser\ '

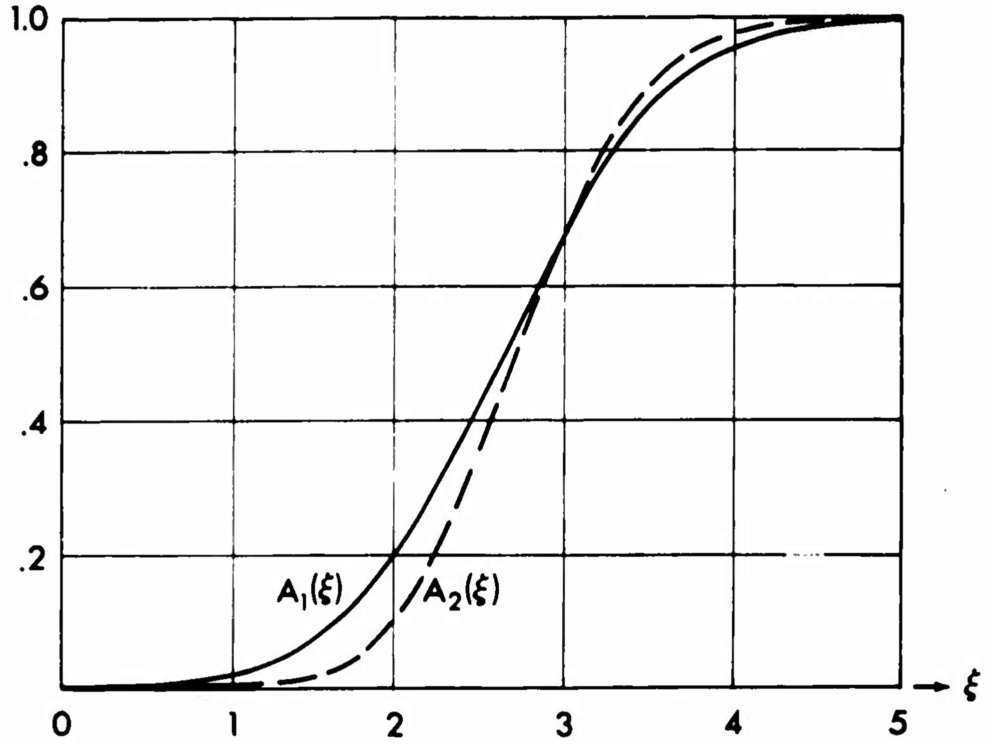

Figure 2

Graphical comparison of the assurance that no failures will occur in a fleet of size $n = 200$, using only the minimum value out of $m = 5$ observations for $A_1(\xi)$, and all five observations for $A_2(\xi)$.

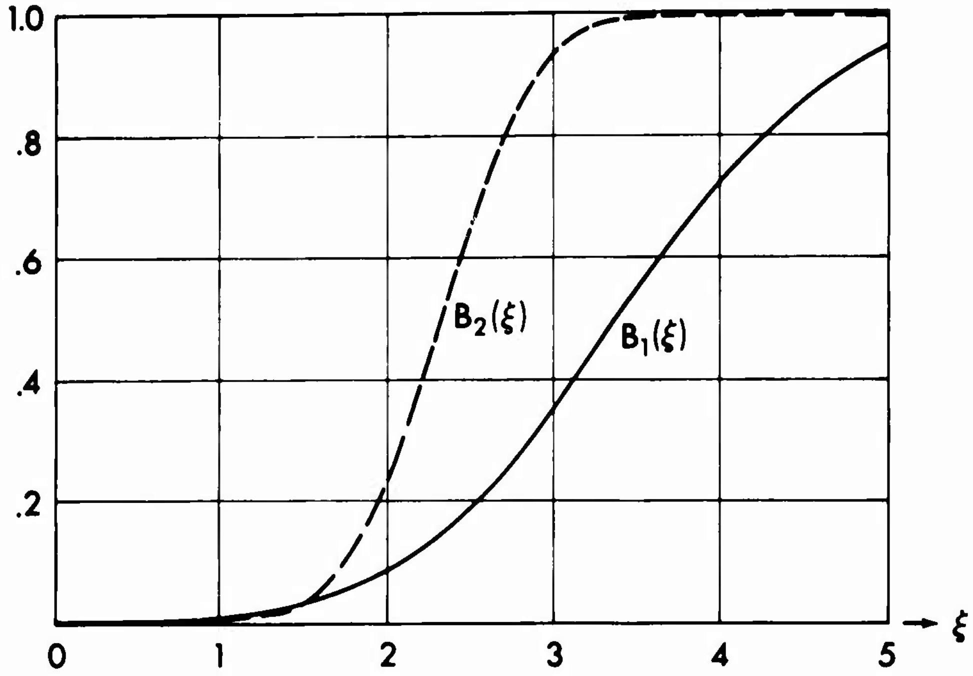

Figure 3

Graphical comparison of the confidence that no more than one percent failures will occur within the future fleet when using only the minimum of $m = 5$ observations for $B_1(\xi)$ with the confidence when using all five observations for $B_2(\xi)$

TABLE I

τ_m the solution of the equation $\frac{\psi(t)}{t} = \frac{1}{m-1}$

m	τ_m	m	τ_m
2	0.506	16	1.641
3	0.765	18	1.693
4	0.936	20	1.740
5	1.062	22	1.781
6	1.160	24	1.819
7	1.241	26	1.853
8	1.309	28	1.884
9	1.368	30	1.912
10	1.420	35	1.975
11	1.466	40	2.029
12	1.508	45	2.076
13	1.545	50	2.117
14	1.580	55	2.153
15	1.611	60	2.187

Interpolation may be used between 15 and 30 to obtain two-decimal-place accuracy.

TABLE II

$$A_2(\xi) = \int_{-\infty}^{\infty} \Phi^n(\xi + \frac{x}{\sqrt{m}})\, d\Phi(x)$$

	n = 200, m = 3	n = 200, m = 5	n = 200, m = 10
ξ	$A_2(\xi)$	$A_2(\xi)$	$A_2(\xi)$
1.00	.005	.001	.000
1.25	.014	.004	.000
1.50	.034	.014	.003
1.75	.074	.042	.017
2.00	.142	.102	.061
2.25	.242	.205	.162
2.50	.369	.349	.326
2.75	.511	.516	.525
3.00	.649	.676	.708
3.25	.768	.805	.843
3.50	.859	.894	.924
3.75	.921	.948	.967
4.00	.959	.976	.987
4.25	.981	.990	.995
4.50	.991	.996	.998
4.75	.996	.999	.999
5.00	.999	.999	1.000

	n = 400, m = 3	n = 400, m = 5	n = 400, m = 10
ξ	$A_2(\xi)$	$A_2(\xi)$	$A_2(\xi)$
1.00	.001	.000	.000
1.25	.005	.001	.000
1.50	.014	.004	.000
1.75	.035	.014	.003
2.00	.077	.043	.016
2.25	.147	.105	.062
2.50	.251	.213	.167
2.75	.382	.363	.341
3.00	.526	.534	.547
3.25	.666	.696	.732
3.50	.784	.822	.862
3.75	.872	.907	.937
4.00	.930	.956	.974
4.25	.965	.981	.990
4.50	.984	.992	.996
4.75	.993	.997	.999
5.00	.997	.999	1.000

TABLE III

$$A_3(\rho) = \int_{-\infty}^{\infty} \Phi^n(\rho+x)\,d\Phi^m(x)$$

ρ	$A_3(\rho)$ n = 400, m = 10	$A_3(\rho)$ n = 400, m = 20	$A_3(\rho)$ n = 400, m = 5
0.	0.024	0.048	0.012
0.25	0.049	0.094	0.025
0.50	0.092	0.170	0.048
0.75	0.160	0.283	0.086
1.00	0.260	0.427	0.145
1.25	0.388	0.587	0.229
1.50	0.533	0.734	0.337
1.75	0.675	0.849	0.463
2.00	0.797	0.924	0.595
2.25	0.886	0.965	0.718
2.50	0.942	0.986	0.820
2.75	0.974	0.995	0.895
3.00	0.989	0.998	0.944
3.25	0.996	0.999	0.973
3.50	0.998	1.000	0.988

ρ	$A_3(\rho)$ n = 200, m = 10	$A_3(\rho)$ n = 200, m = 10	$A_3(\rho)$ n = 200, m = 5
0.	0.048	0.091	0.024
0.25	0.089	0.165	0.047
0.50	0.156	0.273	0.083
0.75	0.252	0.414	0.141
1.00	0.376	0.569	0.222
1.25	0.517	0.716	0.327
1.50	0.658	0.833	0.447
1.75	0.781	0.912	0.580
2.00	0.873	0.958	0.701
2.25	0.933	0.982	0.806
2.50	0.968	0.993	0.883
2.75	0.986	0.997	0.937
3.00	0.994	0.999	0.967
3.25	0.998	1.000	0.985
3.50	0.999	1.000	0.992

REFERENCES

1. Tables of the Binomial Probability Distribution. National Bureau of Standards, U. S. Government (1950).

2. Tables of the Normal Probability Integral, the Normal Density and Its Normalized Derivatives. N.V. Smirnov. Macmillan (1965).